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Heat Diffusion into a Solid Body

Markus Hüning

Fermi National Accelerator Laboratory
Batavia, IL

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Contents

List of Figures

1 Introduction

When high power is applied to an rf cavity the temperature of its inner surface will rise. The temperature increase will be governed by the incident power, the heat capacity and the heat conductivity of the walls. This short note will provide a formula to calculate the increase once these quantities are known.

2 The Equation for Heat Diffusion

The diffusion of heat into a solid body is governed by two quantities, the heap capacity and the heat conductivity. The heat capacity needs to be multiplied with the density of the body since here the specific heat per space is required and not – as common – the specific heat per mass. With these quantities the general equation for heat transfer is

$$\dot{T} = \frac{\lambda}{\rho c} \triangle T + \frac{1}{\rho c} \eta, \tag{1}$$

with λ the heat conductivity [W/(m K)], ρ the density [kg/m^3], and c the specific heat capacity [J/(kg K)]. The quantity η is the dissipated power density [W/m^3]. In absence of an external heat source η the temperature rises if the second spatial derivative of the temperature profile is positive and it falls in the other case. Once a steady heat flow is established, the temperature profile is described by a poisson-type equation.

3 Diffusion after switching off the Source

Here the more easy case without an external source η will be handled. This applies when a certain temperature profile has been established and suddenly the heat source is switched off. Furthermore the 1-dimensional case will be handled. The special example in mind is an quasi-infinite flat body with power being radiated into it from the surface. The diffusion equation simplifies to

$$\dot{T} = \frac{\lambda}{\rho c} T''. \tag{2}$$

A way of solving this equation is by using the Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx f(x) e^{ikx}. \tag{3}$$

The right hand side of the equation 2 turns into an algebraic equation, and the whole thing becomes a regular differential equation of 1st order

$$\dot{\tilde{T}} = -\frac{k^2\lambda}{\rho c}\tilde{T}, \tag{4}$$

which can be solved by

$$\tilde{T}(t,k) = \tilde{T}_0(k) \exp\left(-\frac{k^2\lambda}{\rho c}t\right). \tag{5}$$

The function $\tilde{T}_0(k)$ is defined by the initial condition(s) $\tilde{T}(0,k) = \tilde{T}_0(k)$. The back transformation of this function is

$$T(t,x) = \sqrt{\frac{\rho c}{4\pi\lambda t}} \int\limits_{-\infty}^{\infty} d\xi \, T_0(\xi) \exp\left(-\frac{\rho c}{4\lambda t}(x-\xi)^2\right). \tag{6}$$

Assume the initial temperature profile was a delta function $T_0 = T_{00}\delta(x)$, then the distribution after a while t will be

$$T(t,x) = T_{00}\sqrt{\frac{\rho c}{4\pi\lambda t}} \exp\left(-\frac{\rho c}{4\lambda t}x^2\right). \tag{7}$$

The maximum amplitude of this temperatur profile will decay $\propto 1/\sqrt{t}$, the shape is Gaussian with a σ increasing $\propto \sqrt{t}$. Never mind the pathetic case $t = 0$, that's a delta function.

4 Before switching off the Source

The inhomogenous equation is harder to solve due to missing closed solutions for some of the integrals involved. In Fourier space the solution is simply the solution of an inhomogenious 1st order equation, which can be found by "variation of the constant"

$$\tilde{T}_0(k) \to \tilde{T}_0(t,k). \tag{8}$$

$$\dot{\tilde{T}} = \dot{\tilde{T}}_0 \exp\left(-\frac{k^2\lambda}{\rho c}t\right) - \frac{k^2\lambda}{\rho c}\tilde{T}_0 \exp\left(-\frac{k^2\lambda}{\rho c}t\right) \tag{9}$$

$$=! \quad -\frac{k^2\lambda}{\rho c}\tilde{T}_0 \exp\left(-\frac{k^2\lambda}{\rho c}t\right) + \frac{\tilde{\eta}}{\rho c}, \tag{10}$$

yielding the equation

$$\Leftrightarrow \dot{\tilde{T}}_0 = \frac{\tilde{\eta}}{\rho c} \exp\left(\frac{k^2\lambda}{\rho c}t\right). \tag{11}$$

Assume that the initial condition is $\tilde{T}(0,k) = 0$, the solution is

$$\tilde{T}(t,k) = \frac{\tilde{\eta}}{\lambda k^2}\left[1 - \exp\left(-\frac{k^2\lambda}{\rho c}t\right)\right]. \tag{12}$$

Back transformation of this solution posed some problem for me because I didn't find a nice solution for neither $1/k^2$ nor $\exp(-k^2)/k^2$.

So I decided to find the solution in a different way. Assume the problem is again a dleta-like source in the origin. After an infinitesimal time dt the temperature at the origin will have risen to

$$T = \frac{I}{\rho c}\delta(x)dt. \tag{13}$$

With the use of the delta function I use the intensity I instead of the power density η. What leaves us to do is to sum up all the contributions of the time steps dt , which will be given by equation 7.

$$T(t,x) = \int_0^t d\tau \frac{I}{\sqrt{4\pi\rho c\lambda\tau}} \exp\left(-\frac{\rho c}{4\lambda\tau}x^2\right). \tag{14}$$

This is another nasty integral except for the point $x = 0$.

$$T(t,0) = \int_0^t d\tau \frac{I}{\sqrt{4\pi\rho c\lambda\tau}} = I\sqrt{\frac{t}{\pi\rho c\lambda}}. \tag{15}$$

Note that this is the solution if the heat distributes in both directions. If I assume the (more realistic) case that the heat can diffuse only in one direction, I have to apply a factor of two

$$T(t,0) = I\sqrt{\frac{4t}{\pi\rho c\lambda}}. \tag{16}$$

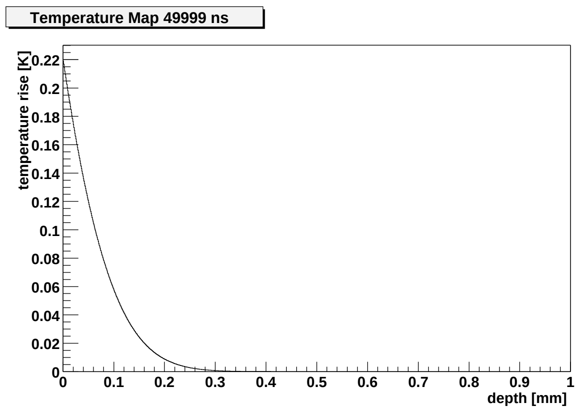

Figure 1: Temperature map of solid copper irradiated with an intensity of 1 MW/m^2 after 50μs at 80 K.

At 200 K Copper has a heat conductivity of 413 W/(m K), a heat capacity of 356 J/(kg K), and a density of 8960 kg/m^3[1]. The temperature rise when applying 1 MW/m^2 of a time of 50 μs is approximately 0.22 K. The width of the temperature profile is harder to estimate due to the lack of a formula. A reasonable guideline may be to take the sigma after 25 μs of free diffusion $\sigma = \sqrt{2\lambda t/(\rho c)} \approx 80\ \mu$m. The figure 1 shows a numerical simulation of the same conditions. The results fit very nicely with the analytic predictions. The figure 2 shows a continuation of the case in figure 1. After 50 μs the heat source was switched off and the heat allowed to diffuse for 150 μs. The result agrees with a gaussian shape as predicted. The predicted σ is 0.23 mm, the maximum height 0.55 K. These numbers agree reasonably with the simulation considering the fact that the two scenarios do not completely agree. The simulated σ is 0.21 mm, the maximum height 0.59 K.

At lower temperatures the heat capacity is considerably reduced whereas the heat conductivity goes up. At 80 K the heat capacity is 201 J/(kg K) and the heat conductivity 557 W/(m K)[2]. Under otherwise identical circumstances the temperature rise increses to 0.25 K after 50 μs of rf. The depth of the temperature profile changes to $\sigma_T \approx 100\,\mu$m.

The table 1 shows the values to be expected in the FNPL rf gun. The

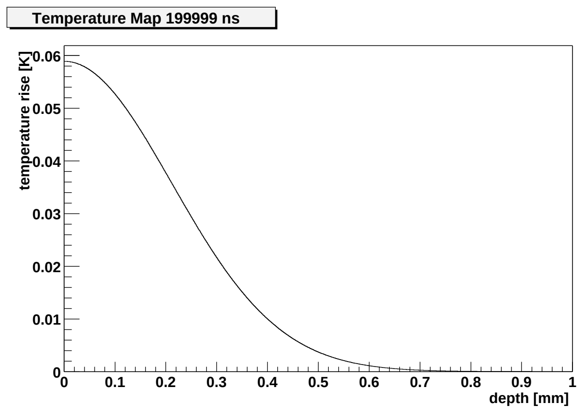

Figure 2: Temperature map of solid copper irradiated with an intensity of 1 MW/m^2 for a duration of 50μs. Afterwards the heat diffused without external source for 150 μs.

peak dissipated power at room temperature is 12.2 MW/m^2. At 80 K this goes down to 4.3 MW/m^2. The corresponding temperature rise is approx. 1 K. The depth of the heated sections is $\sigma_T \approx 100\,\mu$m.

5 Conclusion

The heat dissipation into a solid has been derived. Applying the results to the simulated power dissipation for the FNPL rf gun yields temperature rises of up to 1 K. These will be present for a few milliseconds only before the energy spreads. The outside walls will never see a temperature rise besides that due to the average heat load.

References

[1] David R. Lide (Ed) *Handbook of Chemistry and Physics*, 81st edition, CRC Press, New York, 2000

No	Z_{end} [cm]	R_{end} [cm]	E_{max} [MV/m]	Power [kW]	P/A [kW/cm^2]	$\triangle$T [K]
	0.000	0.00000				
1	0.000	8.91914	35.0000	284.6486	1.1390	32.3
2	5.715	8.91914	0.3997	363.1153	1.1338	32.1
3	6.350	8.28414	2.6791	63.1841	1.1615	32.9
4	6.350	2.75000	39.6594	225.6320	1.1761	33.3
5	7.100	2.00000	43.2164	1.9110	0.1136	3.2
6	7.850	2.75000	43.0464	1.9279	0.1146	3.2
7	7.850	8.28414	39.7899	227.6064	1.1864	33.6
8	8.485	8.91914	2.6940	63.7524	1.1719	33.2
9	18.280	8.91914	0.4080	631.8313	1.1510	32.6
10	18.280	2.65000	38.4996	277.8765	1.2195	34.9
11	19.030	1.90000	42.3345	2.0868	0.1298	3.9
12	25.580	1.90000	12.3523	0.0537	0.0007	0.0
13	30.000	1.90000	0.0024	0.0000	0.0000	0.0
			Total	2143.6261		

Table 1: Results from SUPERFISH on the power dissipation by the rf field in the cavity walls. The peak accelerating field was 35 MV/m the temperature 300 K. This corresponds to a surface resistance of 9.41 mΩ. At 80 K the surface resistance is 3.32 mΩ and hence the power dissipation a factor of 2.8 smaller. The temperature rise has been calculated for T_0 =80 K and after an rf pulse length of 50 μs.

[2] Robert J. Weggel, *A Three-Stage Cryogenic Pulse Magnet Program for BNL Targetry Experiment*, presentation to the Muon Collaboration Technology Board, Feb. 2002